Exercise Sheet 11 due 29 January 2015

1. *Non-degenerate perturbation theory*

Consider an electron in a one-dimensional potential well of width L_z, with infinitely high barriers at $z = 0$ and $z = L_z$. The potential energy inside the potential well is parabolic, of the form $V(z) = u(z - L_z/2)^2$, where u is a real constant. This potential is presumed to be small compared to the energy E_1 of the first confined state of a simple rectangular potential well of the same width L_z.

Find an approximate expression, valid in the limit of small u, for the energy difference between the lowest and first excited states of this well in terms of u, L_z, and fundamental constants.

2. *Wavefunction in perturbation theory*

Consider the perturbation of a non-degenerate state $|n^{(0)}\rangle$. To second order in the perturbation, the perturbed wave function is given by $|n\rangle = |n^{(0)}\rangle + \lambda|n^{(1)}\rangle + \lambda^2|n^{(2)}\rangle + \mathcal{O}(\lambda^3)$. Calculate the energy expectation value $\langle n|\hat{H}_0 + \lambda\hat{H}_1|n\rangle$ to second order in λ. Compare to the expression $E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)}$ derived in the lecture. Where does the discrepancy come from? How is it resolved?

Hint: What is the norm of $|n\rangle$ (to second order)?

3. *Degenerate perturbation theory*

Consider a cubic quantum box for confining an electron. The cube has length L on all three sides, with edges along the x, y, and z directions, and the walls of the box are presumed to correspond to infinitely high potential barriers. We assume that $(x, y, z) = (0,0,0)$ is the point in the *center* of the box.

i. Write down the normalized wavefunctions for the ground state and the first three degenerate excited states for an electron in this box.

ii. Now presume that there is a perturbation $\hat{H}_1 = eFz$ applied (e.g., from an electric field F in the z direction). How does the energy of the ground-state and of the three-fold degenerate excited state change as a result of this perturbation, according to first-order degenerate perturbation theory?

iii. Now presume that a perturbation $\hat{H}_1 = \alpha z^2$ is applied instead. (Such a perturbation could result, e.g., from a uniform fixed background charge density in the box.) Using first-order degenerate perturbation theory, what are the new eigenstates and eigenenergies arising from the three originally degenerate states?

$\int_{-\pi/2}^{\pi/2} \theta^2 \cos^2\theta \, d\theta = \pi^3/24 - \pi/4$ and $\int_{-\pi/2}^{\pi/2} \theta^2 \sin^2 2\theta \, d\theta = \pi^3/24 - \pi/16$